ESTABLISHING A RELATION BETWEEN TURBULENT STRESSES AND THE MEAN FLOW PARAMETERS FOR AN INCOMPRESSIBLE FLUID WITH A POSITIVE PRESSURE GRADIENT

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The isothermal turbulent boundary layer is analyzed which forms during the flow of air through conical or flat diffusers at Reynolds numbers in the 48,850-202,000 range in the entrance. The relation between turbulent stresses and the mean flow parameters in the boundary layer can be established on the basis of the data obtained here.

In order to calculate the friction which appears at a solid surface in a liquid or a gas stream, it is necessary to solve the Navier–Stokes and continuity system of equations with the appropriate constraints. This is a system of second-order nonlinear differential equations and, therefore, a mathematical solution is very difficult here.

An effective method of solving this problem is to replace the system of equations by the Prandtl boundary layer equations. The latter, in conjunction with the continuity equation, make it rather easy to solve the friction problem for a laminar flow. When the flow is turbulent, however, difficulties arise in the treatment of the boundary layer equations as a result of not knowing how the tangential and the normal stresses are related to the mean flow parameters.

In recent years one has often solved the problem by applying the momentum integral equation (Karman equation) to the boundary layer:

$$\frac{d\theta}{dx} = -(H+2)\frac{\theta}{u_1}\cdot\frac{du_1}{dx} + \frac{c_f}{2} + \frac{1}{u_1^2}\cdot\frac{d}{dx}\int_0^\infty (\overline{u'^2} - \overline{v'^2})\,dy. \tag{1}$$

To this very day there is no unanimous opinion as to the significance of the term

$$I = \frac{1}{u_1^2}\cdot\frac{d}{dx}\int_0^\infty (\overline{u'^2} - \overline{v'^2})\,dy. \tag{2}$$

This must be attributed to the lack of sufficient experimental data.

We will analyze here the isothermal turbulent boundary layer which forms during the flow of air through a conical of a flat diffuser. The study was made using an aerodynamic duct of the open kind. The tests covered a range of Reynolds numbers from 48,500 to 202,000 in the entrance (see Table 1).

The conical diffusers were made 500 mm long with 100 mm entrance diameters and an 8° or a 10° divergence angle. The important variables were measured at sections 0, 30, 75, 135, 202, and 360 mm away from the entrance. The flat diffuser had a 40 × 180 mm cross section at the entrance and was 174 mm long. The lower and the upper wall were movable so that the divergence angle could be varied. Tests were performed here with a 10°, 12°, and 14° divergence angle. In all cases the important variables were measured at sections 0, 30, 60, 90, 130, and 170 mm away from the entrance.

At each section we measured the average velocity profile, the turbulence of the axial and the normal velocity components, the turbulent shearing stress profile, and the correlation between axial and normal velocity pulsations at a test point.

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TABLE 1. Velocities of a Potential Flow and Corresponding Reynolds Numbers in the Entrance Sections of Diffusers

Conical diffuser		Flat diffuser	
divergence angle, deg	Reynolds number	divergence angle, deg	Reynolds number
8	48500 135500 200000	10	58200
		12	57600
10	52700 145700 202000		
		14	58800

The measurements were performed with a UTA-5B electrothermoanemometer fully described in [1].

On the basis of the test data, an attempt is made to establish the relation between turbulent stresses and the mean flow parameters.

For convenience, the parameters in Eq. (2) are put in dimensionless form.

The integral (2) can be written as

$$I = \frac{1}{\bar{u}_1^2} \cdot \frac{d}{d\bar{x}} \bar{u}_1^2 \frac{1}{d_{\text{equ}}} \int_0^\delta (\varepsilon_u^2 - \varepsilon_v^2)\, dy, \tag{3}$$

where $\bar{u}_1 = u_1/u_0$ is the dimensionless velocity at the edge of the boundary layer, $\bar{x} = x/d_{\text{equ}}$ is the dimensionless coordinate, and d_{equ} is the equivalent diffuser diameter ($d_{\text{equ}} = 100$ mm for the conical diffusers and $d_{\text{equ}} = 65.4$ mm for the flat diffuser).

The values of

$$\bar{L} = \frac{1}{d_{\text{equ}}} \int_0^\delta (\varepsilon_u^2 - \varepsilon_v^2)\, dy \tag{4}$$

were computed for each section of the boundary layer under all flow conditions. An analysis of the results has yielded the following relation

$$\bar{L} = c\bar{\theta}H^m,$$

where $\bar{\theta} = \theta/d_{\text{equ}}$ is the dimensionless momentum thickness and c and m are constants.

In order to determine the exponent in this relation, the test data have been plotted on a graph to a log – log scale (Fig. 1). The test points are seen to lie close to a straight line with the slope m = 3. The constant c is equal to 0.005. Thus, the turbulent stresses may be related to the mean flow parameters as follows:

$$I = \frac{0.005}{u_1^2} \cdot \frac{d}{dx}(\theta H^3 u_1^2). \tag{5}$$

We will note that D. Ross [3] derived a relation between Reynolds stresses and the mean flow parameters on the basis of the P. Granville hypothesis that

$$\int_0^\infty (\overline{u'^2} - \overline{v'^2})\, dy \sim \delta^*.$$

Using the data obtained by G. B. Schubauer and P. S. Klebanoff [2] and by J. Laufer [4], D. Ross [3] could derive the following kind of relation between turbulent stresses and the mean flow parameters:

$$I = \frac{2.67}{u_1^2} \cdot \frac{d}{dx}(0.006\delta^* u_1^2). \tag{6}$$

The integral term on the right-hand side of Eq. (1) will be evaluated here according to formulas (5) and (6), on the basis of the experimental data given in [2]. The calculated results are shown in Fig. 2 in dimensionless coordinates. The unit length is l = 3.048 m and the unit velocity is the velocity at the x = 5.38 m section, where flow begins under $dP/dx > 0$. The form factor H and the velocity distribution along the edge of the boundary layer have been assumed known from the data in [2]. The coefficient of skin friction c_f has been calculated by the Ludwig–Tilman formula:

$$c_f = 0.246 \left(\frac{u_1\theta}{\nu} \right)^{-0.268} 10^{-0.678H}.$$

It can be seen from Fig. 2 that both formulas (5) and (6) yield similar values for the integral I, with the Ross formula (curve 4) giving somewhat lower values than formula (5) (curve 3).

The evaluation of the terms in Eq. (1) has confirmed that, as separation is approached, the usually disregarded last term on the right-hand side of Eq. (1) becomes quite large and near separation becomes comparable with the first term. Therefore, the skin friction coefficient, which is obtained from the

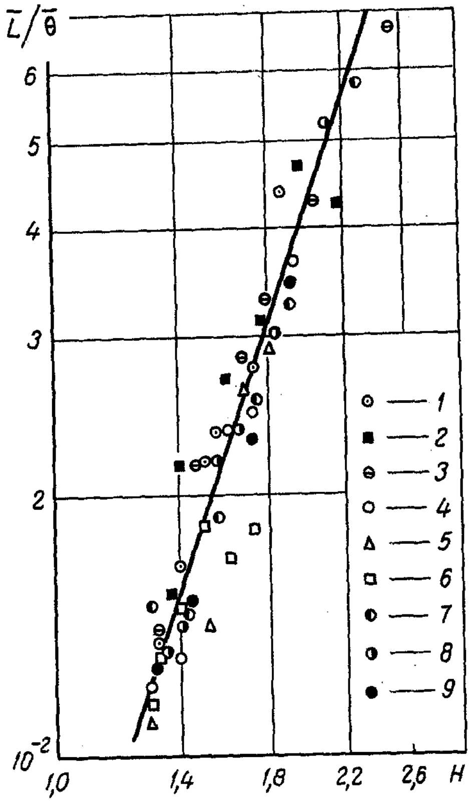

Fig. 1

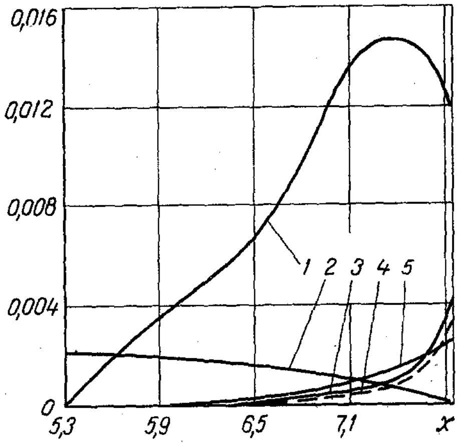

Fig. 2

Fig. 1. The integral term $\overline{L}/\overline{\theta}$ as a function of the form factor H: flat diffuser with a 10° (1), a 12° (2), and 14° (3) divergence angle; conical diffuser with an 8° divergence angle and with a Reynolds number 48,500 (4), 135,500 (5), and 200,000 (6) in the entrance; conical diffuser with a 10° divergence angle and with a Reynolds number 52,700 (7), 145,700 (8), and 202,000 (9) in the entrance.

Fig. 2. Values of individual terms on the right-hand side of Eq. (1): $1(H+2)\,\theta/u_1 \cdot du_1/dx$ (1), $c_f/2 = 0.123 \cdot 10^{-0.678H}\, Re_\theta^{\,0.268}$ (2) $(1/u_1^2 \cdot d/dx)\int_0^\infty (\overline{u}'^2 - v'^2)\, dy$ (3-5), with the approximations according to Eqs. (5), (6), and (20), respectively (x,m).

momentum equation with the integral term omitted, increases as separation is approached and this contradicts the experimental results where the values of c_f have been determined from thermoanemometer measurements.

Inserting (5) into (1), we obtain the Karman momentum equation:

$$\frac{d\theta}{dx} = -(H+2)\frac{\theta}{u_1} \cdot \frac{du_1}{dx} + \frac{c_f}{2} + \frac{0{,}005}{u_1^2} \cdot \frac{d}{dx}(\theta H^3 u_1^2) \tag{7}$$

or

$$\frac{d\theta}{dx} = \frac{\frac{c_f}{2} - (H + 2 - 0{,}01H^3)\frac{\theta}{u_1} \cdot \frac{du_1}{dx} + 0{,}015\theta H^2 \frac{dH}{dx}}{(1 - 0{,}005H^3)}. \tag{8}$$

It would be of interest to study the change in momentum thickness calculated from the momentum equation with and without the turbulence terms taken into account. This is best done using the experimental data by Schubauer and Klebanoff.

The results of calculations are shown in Fig. 3, where the axial distribution of momentum thickness (as in Fig. 2, $l = 3.048$ m) along the wing profile length is plotted from the x = 5.38 section with dP/dx > 0 on. It can be seen here that the calculated I-curve, which includes the turbulence terms in the near-separation regions, does not bend downward as does curve 2 calculated disregarding the turbulence terms, and this indicates the considerable effect of the turbulence components on the momentum thickness. The

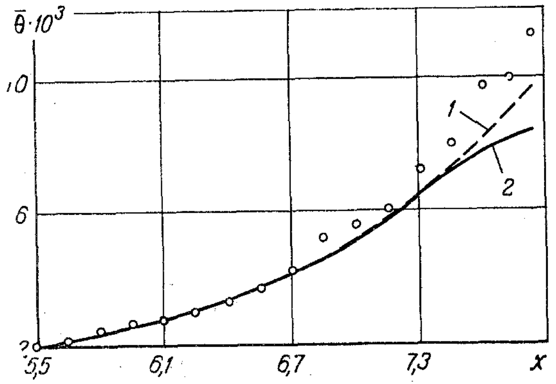

Fig. 3

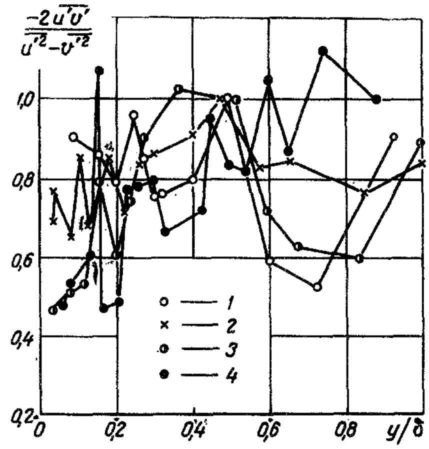

Fig. 4

Fig. 3. Effect of normal stresses on the magnitude of θ: according to Eq. (1), according to the Karman momentum equations (2). The points on the graph correspond to measured values of momentum thickness.

Fig. 4. Relation between normal and tangential stresses in a boundary layer: flat diffuser with a 12° divergence angle and Re = 57,600 at $\bar{x} = 0.46$ (1) and $\bar{x} = 2.6$ (2); conical diffuser with a 10° divergence angle, Re = 145,700 at $\bar{x} = 2.8$ (3); according to Schubauer–Klebanoff measurements [2] (4) at x = 7.46 m. $-\theta/u_1\, du_1/dx = 0.0034$ (1), 0.0037 (2), 0.0036 (3), and 0.0037 (4).

measured values of momentum thickness, according to the data in [2], lie somewhat above the calculated curve as a result, apparently, of secondary flow modes produced in the wide 3.048 m long test segment of the Schubauer–Klebanoff profile.

Thus, as Figs. 2 and 3 indicate, the integral term in Eq. (1) must be taken into account when the momentum equation is used for calculating the boundary layer in a turbulent flow under a positive pressure gradient. Disregarding this term would give rise to significant errors in the determination of the friction coefficient and of the mixing length near separation.

Another approach to solving the problem is to express the normal stresses in the integral term (2) in terms of tangential stresses:

$$(\overline{u'^2} - \overline{v'^2}) = f(\overline{u'v'}).$$

For this purpose, the local values of the tangential stress to normal stress ratio

$$\varkappa = \frac{-2\overline{u'v'}}{\overline{u'^2} - \overline{v'^2}} \tag{9}$$

are calculated from the pulsation intensity of the axial and the normal velocity components and from the turbulent shearing stress profile at the various boundary layer sections. In Fig. 4 is shown the distribution of $\varkappa$ values across the boundary layer thickness at different sections of conical and flat diffusers, these values having been obtained from our measurements and from [2] at approximately the same values of $-\theta/u_1\, du_1/dx$. It is evident here that the magnitude of $\varkappa$ fluctuates considerably about some mean value, while the latter definitely tends to increase as the outer edge of the boundary layer is approached.

In Fig. 5 are shown the mean values of $\varkappa$ at various boundary layer sections, as a function of $-\theta/u_1 \cdot du_1/dx$, for different Reynolds numbers in the entrance section of a flat and a conical diffuser. On the same diagram are shown the corresponding values measured by Schubauer and Klebanoff [2] and by Laufer [4], the structure of a turbulent boundary layer in a flat channel having been thoroughly studied in [4]. The distribution of test points in Fig. 2 based on measurements made in [2] and [4] does not provide sufficient evidence for firm conclusions. An overall analysis of all points in Fig. 2 may, on the other hand, yield an approximate relation:

$$\varkappa = b - k\frac{\theta}{u_1} \cdot \frac{du_1}{dx}. \tag{10}$$

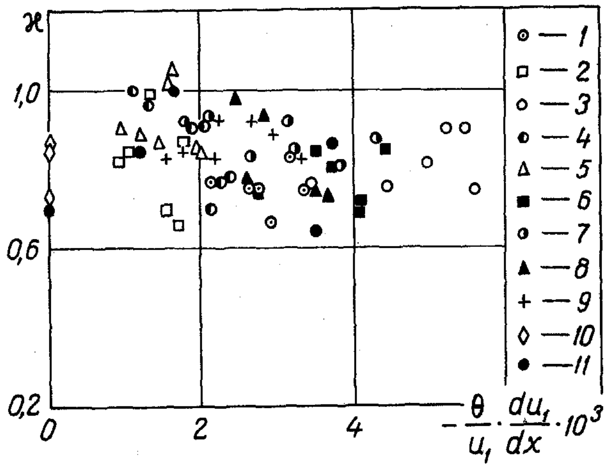

Fig. 5. Parameter ϰ as a function of $-\theta/u_1 \cdot du_1/dx$: flat diffusers with a 10° (1), 12° (2), and 14° (3) divergence angle; conical diffuser with an 8° divergence angle and Reynolds number 45,800 (4), 135,500 (5), and 200,000 (6); conical diffuser with a 10° divergence angle and Reynolds number 52,700 (7), 145,700 (8), and 202,000 (9) in the entrance section; according to measurements in [4] (10); according to measurements in [2] (11).

The values of coefficients k and b are determined by the method of least squares: $k = -19.60$ and $b = 0.883$. The straight line representing Eq. (10) approximates the test data with a ±18.6% maximum deviation of test points for ϰ.

In this way,

$$\frac{-2\overline{u'v'}}{\overline{u'^2} - \overline{v'^2}} = 0.883 + 19.6\frac{\theta}{u_1}\cdot\frac{du_1}{dx} \tag{11}$$

or

$$\overline{u'^2} - \overline{v'^2} = \frac{-2.27\overline{u'v'}}{\left(1 + 22.2\frac{\theta}{u_1}\cdot\frac{du_1}{dx}\right)}. \tag{12}$$

If the tangential shearing stresses in the turbulent flow field are known, then the normal stresses can be calculated from Eq. (12).

We will note that Ross [3] used the anisotropy of turbulent stresses for expressing the relation between normal and tangential stresses as

$$\overline{u'^2} - \overline{v'^2} = -\frac{2\overline{u'v'}}{\operatorname{tg} 2\alpha}, \tag{13}$$

where α is the angle between the principal axis of the turbulent stress tensor and the direction of mean flow. H. L. Dryden has suggested in [5] that the turbulent shearing stress may be related to the local anisotropic turbulence and that $\tan 2\alpha$ may be assumed approximately constant. D. Ross has confirmed this hypothesis and, analyzing the data in [2] and [4], has found that for purposes of calculation one may assume $\tan 2\alpha$ to be constant and approximately equal to 0.75, i.e., that

$$\overline{u'^2} - \overline{v'^2} = -2.67\overline{u'v'}. \tag{14}$$

In analyzing the test data in [2] and [4], Ross used the pulsation components of velocity u' and v' at the boundary layer section where $dP/dx = 0$. This explains why the value of ϰ obtained by him is independent of the pressure gradient.

An analysis of the velocity pulsations measured by Schubauer and Klebanoff at boundary layer sections where the pressure gradient is positive will show that the tangential stress to normal stress ratio decreases somewhat as the pressure gradient increases. These data can be sufficiently well approximated by the equation

$$\varkappa = 0.81 + 4.14 \frac{\theta}{u_1} \cdot \frac{du_1}{dx}. \tag{15}$$

A comparison between Eqs. (12) and (14) with $du_1/dx = 0$ indicates that they yield close results.

If Eq. (12) is inserted into Eq. (2), we obtain

$$I = \frac{2.27}{u_1^2} \cdot \frac{d}{dx}\left[\frac{\int\limits_0^\delta -\overline{u'v'}\,dy}{\left(1 + 22.2\frac{\theta}{u_1} \cdot \frac{du_1}{dx}\right)}\right]. \tag{16}$$

Thus, under the integral in (16) there appear tangential stresses whose values can be taken from tests.

K. K. Fedyaevskii [6] has represented the tangential shearing stress distribution across a turbulent boundary layer section in terms of a fourth-degree polynomial in $\eta = y/\delta$. With the boundary conditions taken into account, this polynomial is

$$\frac{\tau}{\rho u_1^2} = \frac{\tau_w}{\rho u_1^2}(1 - 4\eta^3 + 3\eta^4) - \frac{\delta}{u_1} \cdot \frac{du_1}{dx}(\eta - 3\eta^3 + 2\eta^4). \tag{17}$$

This equation does not account for the effect of the profile form factor on τ, and this form factor varies considerably with the pressure gradient. Moreover, it is convenient to replace the boundary layer thickness in (17) by the momentum thickness θ.

Based on a generalization of experimental data on turbulence and on the mean values of boundary layer parameters for a flow of an incompressible fluid under a positive pressure gradient [1], we have established the distribution of tangential shearing stresses as:

$$\frac{\tau}{\rho u_1^2} = \frac{c_f}{2} \cdot f_1(\eta) - \frac{\theta}{u_1} \cdot \frac{du_1}{dx}\left(\frac{H - 1.33}{0.25H}\right) f_2(\eta), \tag{18}$$

where

$$f_1(\eta) = 1 - 2{,}1\eta^{1,1} + 1{,}1\eta^{2,1};$$

$$f_2(\eta) = \begin{cases} 6.66\eta - 11.11\eta^3 & \text{for} \quad 0 \leqslant \eta \leqslant 0.3; \\ 11.69(1-\eta)^3 - 12.53(1-\eta)^4 & \text{for} \quad 0.3 \leqslant \eta \leqslant 1. \end{cases}$$

Therefore, expressing the tangential stresses according to (18) and inserting these expressions into (16) will yield

$$I = \frac{2.27}{u_1^2} \cdot \frac{d}{dx}\left\{\frac{u_1^2 \int\limits_0^\delta \left[\frac{c_f}{2} f_1(\eta) - \frac{\theta}{u_1} \cdot \frac{du_1}{dx}\left(\frac{H - 1.33}{0.25H}\right) f_2(\eta)\right] dy}{\left(1 + 22.2\frac{\theta}{u_1} \cdot \frac{du_1}{dx}\right)}\right\}. \tag{19}$$

Changing y to the variable of integration y/δ and assuming that $\delta \approx 8\theta$ on the average, we can obtain a following expression for the integral in (19):

$$I = \frac{18.2}{u_1^2} \cdot \frac{d}{dx}\left\{\frac{\theta u_1^2\left[0.1775 c_f - 0.48\frac{\theta}{u_1} \cdot \frac{du_1}{dx}\left(\frac{H - 1.33}{0.25H}\right)\right]}{\left(1 + 22.2\frac{\theta}{u_1} \cdot \frac{du_1}{dx}\right)}\right\}. \tag{20}$$

Such a representation of the integral term (2) in the momentum equation (1) facilitates its numerical evaluation. This integral term expressed as in (20) has been computed from the test data in [2]. The result of this computation is shown in Fig. 2 (curve 5) for a comparison with the results obtained by formulas (5) and (6). It is evident that formula (20) yields almost the same value for the integral I as formulas (5) and (6), but the authors believe that the method of calculating the integral in the form (5) is somewhat simpler.

NOTATION

x	is the axial distance along a diffuser, measured in the direction of flow;
y	is the distance along the normal to a diffuser, measured from the surface;
u_1	is the mean velocity at the edge of a boundary layer;
u_0	is the mean velocity at the edge of a boundary layer in the diffuser entrance;
δ	is the boundary layer thickness;
δ^*	is the boundary layer displacement thickness;
θ	is the momentum thickness of a boundary layer;
H	is the form factor of the velocity profile across a boundary layer δ^*/θ;
$\varepsilon_u = (\sqrt{\overline{u'^2}}/u_1)$, $\varepsilon_v = (\sqrt{\overline{v'^2}}/u_1)$	are the turbulence intensity components of axial velocity;
$\rho\overline{u'v'}$, $\rho\overline{u'^2}$, $\rho\overline{v'^2}$	are the tengential and normal stresses.

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